A note on an invariant sum problem in geometry

by

Zhao, D.S., Lee, T. Y., Yap, S.F., Lee, C.S.

Mathematics and Mathematics Education Academic Group
National Institute of Education, Nanyang Technological University,
1, Nanyang Walk, Singapore 637616

In this note we consider the problem of finding the perimeter sum total of a finite number of similar geometrical figures.

Equilateral triangles

Consider a finite number of k equilateral triangles of different sizes which are arranged with their bases lying on a line segment AB of fixed length l as shown in Figure 1. Do you expect that the sum of the perimeters of these k equilateral triangles is independent of the number k as well as the different side lengths? Let's find out. Let $l_1, l_2, \cdots, l_k$ represent, respectively, the side lengths of the k equilateral triangles arranged in order from A to B. Then $l = \sum_{i=1}^{k} l_i$. It follows that the sum of the perimeters of the k equilateral triangles equals $\sum_{i=1}^{k} 3 \cdot l_i = 3 \cdot l$. Hence the sum only depends on the length l of AB!

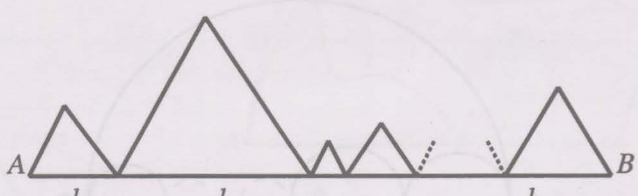

Figure 1

It is interesting to note that the above fact holds true not only for equilateral triangles, it is in fact true for other similar figures, as will be shown below. Some elementary notions in calculus will be used in the proof. We leave it as an exercise to readers unfamiliar with calculus to show the following two cases.

Exercise 1. Consider k triangles which are all similar to ΔABC, and are of various sizes, arranged on a line segment AB as shown in Figure 2. Show that the sum of the perimeters of the k similar triangles is equal to the perimeter of ΔABC.

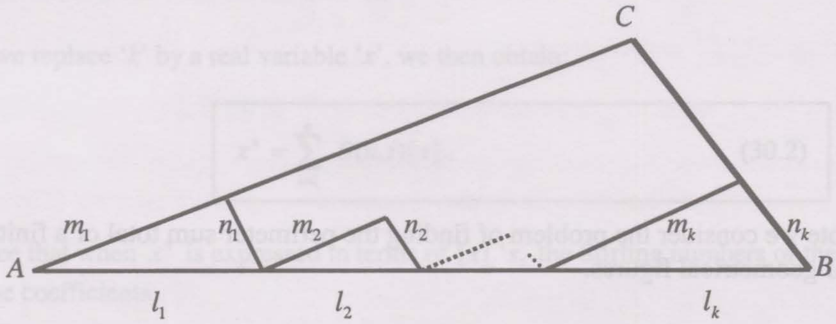

Figure 2

Exercise 2.

Consider k circles of various sizes and with centres all lying on a line segment AB as shown in Figure 3. Show that the sum of the circumferences of the k circles is equal to the circumference of the circle with diameter AB. (Thus the sum is once again independent of k and the diameter lengths of the k circles.)

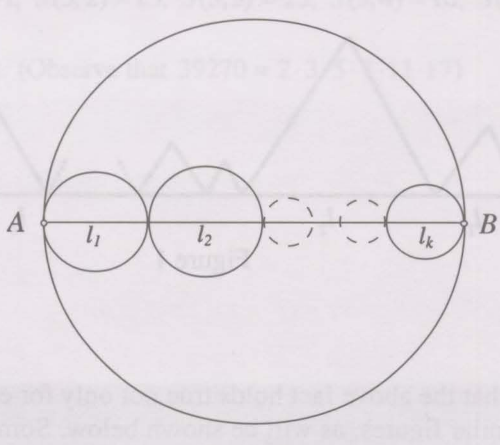

Figure 3

A general case

Let us now generalize the above observations. Consider a finite number of k similar curves as shown in Figure 4. Each of the k curves is similar to the biggest one passing through A and B. All the k similar curves are arranged with one following another on the line segment AB. Let l be the length of AB, and let $l_1, l_2, \cdots, l_k$ be the lengths of the respective sides of the k similar curves which are coincident with AB. In order to calculate the arc length of the i^{th} curve, we position it at A as shown in Figure 5.

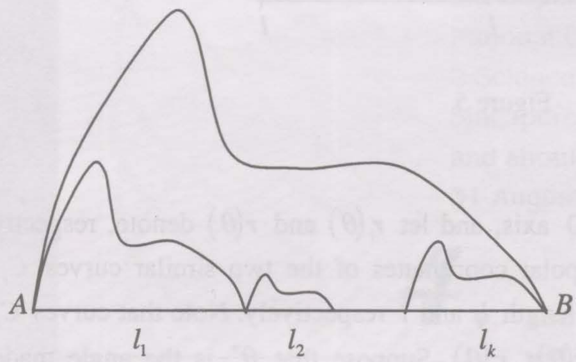

Figure 4

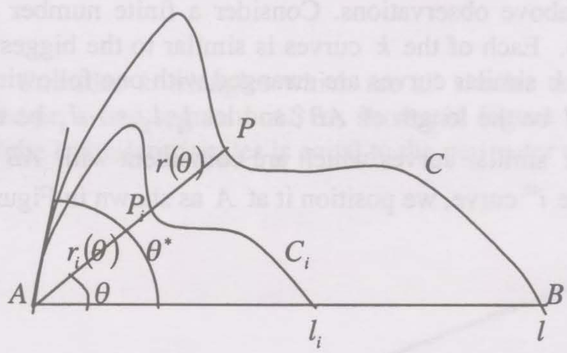

Figure 5

Let AB be placed on the $\theta = 0$ axis, and let $r_i(\theta)$ and $r(\theta)$ denote, respectively, the distance functions from A in polar coordinates of the two similar curves C_i and C whose overlapping bases are of length l_i and l respectively. Note that curves C_i and C are similar means that $l : l_i = r(\theta) : r_i(\theta)$. Suppose that θ^* is the angle made by the tangent to the curve APB at A with AB, then the arc length of the i^{th} curve equals

$$l_i + \int_{\theta=0}^{\theta=\theta^*} r_i(\theta) d\theta = l_i + \int_{\theta=0}^{\theta=\theta^*} \tfrac{l_i}{l} r(\theta) d\theta .$$

Therefore, the sum of the arc length of all the k similar closed curves equals

$$\sum_{i=1}^{k} l_i + \sum_{i=1}^{k} \int_{\theta=0}^{\theta=\theta^*} r_i(\theta) d\theta = l + \tfrac{1}{l} \sum_{i=1}^{k} \int_{\theta=0}^{\theta=\theta^*} l_i \cdot r(\theta) d\theta = l + \tfrac{1}{l} \sum_{i=1}^{k} l_i \int_{\theta=0}^{\theta=\theta^*} r(\theta) d\theta = l + \int_{\theta=0}^{\theta=\theta^*} r(\theta) d\theta$$

which is the arc length of the closed curve C.

Conclusion

Our objective in the above investigation of problems involving similar figures is to motivate students in their learning of geometry. Readers may like to begin with finding out in what ways the above result could be further generalized.